Note on the swimming of slender fish

By M. J. LIGHTHILL
Royal Aircraft Establishment, Farnborough, Hampshire

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The paper seeks to determine what transverse oscillatory movements a slender fish can make which will give it a high Froude propulsive efficiency,

$$\frac{(\text{forward velocity}) \times (\text{thrust available to overcome frictional drag})}{(\text{work done to produce both thrust and vortex wake})}.$$

The recommended procedure is for the fish to pass a wave down its body at a speed of around $\frac{5}{4}$ of the desired swimming speed, the amplitude increasing from zero over the front portion to a maximum at the tail, whose span should exceed a certain critical value, and the waveform including both a positive and a negative phase so that angular recoil is minimized. The Appendix gives a review of slender-body theory for deformable bodies.

1. Introduction

It has frequently been noted that swimming speeds attained both by fish and by mammals (such as porpoises) are remarkably high in relation to their available muscle power. This has been held to indicate, first, that frictional drag is so low that a large fraction of the boundary layer must be laminar, and, secondly, that the movements generating the thrust to balance frictional drag must produce only a small fractional drag increase as a result of the vorticity they create. Such achievements should be regarded as a challenge to other species wishing to propel themselves economically through fluids.

The analysis of both these postulated achievements must be helped if the flow outside the boundary layer—that is, the inviscid-fluid flow around the swimming animal—can be determined. This provides the environment in which the boundary layer develops, and determines how and how much vorticity is shed.

Now this external flow is easy to evaluate approximately for what we shall call a 'slender fish'—namely, either a fish or a swimming mammal, whose dimensions and movements at right angles to its direction of locomotion are small compared with its length, while its cross-section varies along it only gradually. The 'slender-body' theory (which goes back to Munk's work (1924) on flow about airships; for a general review see Lighthill (1960)) can be applied to these.

In this note the theory is worked out for a slender fish which makes swimming movements only in a single direction at right angles to its direction of locomotion. Types of movement producing thrust with a small vortex-drag penalty are determined, and some remarks made on the boundary-layer development which they might induce.

2. Inviscid-flow theory of the forces produced by swimming movements

It is convenient to consider swimming movements which enable the fish to stay still in water flowing with velocity U in the x-direction. We investigate the inviscid flow around the fish, remembering that, for the calculation to be relevant, it must yield a mean thrust on the fish sufficient to balance the mean frictional drag.

We describe the fish in the stream as 'stretched straight' when it is held stationary in a standard position such that no resultant normal force acts on any cross-section. When the fish is stretched straight the cross-section of its surface S at a distance x downstream from the nose will be denoted by S_x.

We suppose now that in swimming the cross-section S_x receives a displacement $h(x,t)$ from the stretched straight position, in the z-direction, so that the displacement is at right angles to the direction of locomotion and varies both with position and time. Then on slender-body theory the flow can be regarded as compounded of

(i) the steady flow around the stretched straight body;
(ii) the flow due to the displacements $h(x,t)$.

To envisage the flow component (ii), we observe that a cross-section S_x moves, relative to the fluid flowing past it with velocity U, at a velocity

$$V(x,t) = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}, \tag{1}$$

and that, locally, the body shape differs little from that of an infinite cylinder C_x whose cross-section is S_x all the way along. Accordingly, to the slender-body approximation, the flow component (ii) near S_x is identical with the two-dimensional potential flow that would result from the motion of the cylinder C_x through fluid at rest with velocity $V(x,t)$.

We suppose now that this flow has momentum

$$\rho V(x,t)\,A(x) \tag{2}$$

per unit length of cylinder, where ρ is the density. In the usual terminology, $\rho A(x)$ is the 'virtual mass' of the cylinder C_x per unit length for motions in the z-direction. Thus, the coefficient $A(x)$ has the dimensions of area; for example, it is equal to the area of the cross-section S_x when the latter is circular, while for an ellipse with minor axis in the z-direction $A(x)$ is the area of its circumscribing circle.

To obtain the instantaneous lift per unit length of fish, $L(x,t)$, that is, the force in the z-direction on the cross-section S_x, we observe that this must be equal and opposite to the rate of change of momentum of the fluid passing S_x; that is,

$$L(x,t) = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\{V(x,t)\,A(x)\}. \tag{3}$$

We next write down the rate, W, at which the fish does work by making displacements $h(x,t)$ in the direction in which these lift forces act; this is

$$\begin{aligned} W &= -\int_0^l \frac{\partial h}{\partial t} L(x,t)\,dx = \rho\int_0^l \frac{\partial h}{\partial t}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\{V(x,t)\,A(x)\}\,dx \\ &= \rho\int_0^l \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left\{\frac{\partial h}{\partial t}\,V(x,t)\,A(x)\right\}dx - \rho\int_0^l \frac{\partial V}{\partial t}\,VA(x)\,dx \\ &= \frac{\partial}{\partial t}\left\{\rho\int_0^l \frac{\partial h}{\partial t}\,VA(x)\,dx - \tfrac{1}{2}\rho\int_0^l V^2A(x)\,dx\right\} + \rho U\left[\frac{\partial h}{\partial t}\,VA(x)\right]_0^l. \end{aligned} \tag{4}$$

The mean over a long time of the time-derivative in this last expression is zero. As regards the final term, we have $A(0) = 0$, but because the fish has a tail $A(l)$ is non-zero—at least on the approximation of assuming a straight trailing edge, which makes $A(l)$ equal to the area of the circle with the trailing edge as diameter. Hence, (4) gives for the mean rate of working by the fish

$$\overline{W} = \rho UA(l)\overline{\left\{\frac{\partial h}{\partial t}\left(\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\right)\right\}}_{x=l}. \tag{5}$$

This value can be interpreted physically as the mean of the product of the lateral velocity $\partial h/\partial t$ of the tail trailing edge with the rate of shedding $(\rho VA)\,U$ of lateral momentum (2) behind the trailing edge, on the argument that rate of working equals velocity times rate of change of momentum.

On the other hand, the rate of shedding of kinetic energy of lateral fluid motions is $(\frac{1}{2}\rho V^2A)\,U$, with (again) the trailing edge values of V and A. If, now, we subtract the mean value of this from the mean rate of working $\overline{W}$, we should obtain the rate of working available for producing the mean thrust, say, $\overline{P}$; this rate is $\overline{P}U$. If the result of this subtraction were negative, one would infer that there was a mean drag, more energy being shed in the wake than is put into the fluid by the swimming movements. In this case an external force would be needed to supply the extra work to push the fish through the fluid against this drag and the frictional drag together. *Positive* thrust, on the other hand, would produce motion at the speed at which it is exactly balanced by frictional drag.

To sum up,

$$\overline{W} - \tfrac{1}{2}\rho\overline{V^2}AU = \overline{P}U, \tag{6}$$

whence by (5) the mean thrust is

$$\overline{P} = \tfrac{1}{2}\rho A(l)\left\{\overline{\left(\frac{\partial h}{\partial t}\right)^2} - U^2\overline{\left(\frac{\partial h}{\partial x}\right)^2}\right\}_{x=l}. \tag{7}$$

The above arguments for determining $\overline{P}$ are attractively simple, but may not be overwhelmingly convincing. To put it beyond a doubt that the inviscid-flow slender-body pressure distributions do yield (7) as mean resultant thrust, the author has calculated them in the Appendix (which was in any case desirable) and demonstrated the point—and may as well admit that he thought of the above arguments only after getting the answer by the method of the Appendix.

3. Conditions for efficient thrust

The dynamic problem of obtaining efficient thrust, with the values of forces which were obtained in § 2, is as follows. We want a high value of the inviscid-flow efficiency of the swimming movements, which by (5) and (6) is

$$\eta_F = \frac{\overline{P}U}{\overline{W}} = 1 - \frac{1}{2}\left\{\overline{\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)^2}\right\}_{x=l} \Big/ \left\{\overline{\frac{\partial h}{\partial t}\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)}\right\}_{x=l}. \tag{8}$$

The suffix F here is intended to suggest 'Froude efficiency', which similarly relates thrust to the total kinetic energy added to the fluid—including energy devoted to forming a vortex wake.

At the same time the displacements $h(x,t)$ which the fish can make are restricted by two considerations. First, the rate of change of lateral momentum which they produce must be equal to the resultant of the lift forces; this means that

$$\rho\int_0^l S(x)\frac{\partial^2 h}{\partial t^2}dx = -\rho\int_0^l \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{A(x)\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)\right\}dx, \tag{9}$$

where $S(x)$ is the area of the cross-section S_x and the density of the fish has been taken equal to the density ρ of the water. Secondly, the rate of change of angular momentum about the y-axis which they produce must be equal to the moment of the lift forces about that axis:

$$\rho\int_0^l xS(x)\frac{\partial^2 h}{\partial t^2}dx = -\rho\int_0^l x\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{A(x)\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)\right\}dx. \tag{10}$$

Any movements attempted by the fish which failed to satisfy (9) and (10) would automatically produce reactions, or 'recoils', in the form of rigid-body movements

$$F(t) + xG(t), \tag{11}$$

which when added to $h(x,t)$ would cause it to satisfy (9) and (10).

Now, from (8), we see that an efficiency η_F only a little less than 1 will be obtained if at the trailing edge typical values of $V = \partial h/\partial t + U\,\partial h/\partial x$ are small compared with those of $\partial h/\partial t$, but positively correlated with them. The last point is essential; negative correlation would produce negative thrust. The first point, on the other hand, must not be overdone: although for given tail velocity $\partial h/\partial t$ the efficiency is increased by making $\partial h/\partial t + U\,\partial h/\partial x$ smaller, we see from (7) that the actual thrust is reduced, so that a compromise is required.

The condition that $\partial h/\partial t + U\,\partial h/\partial x$ must have values generally smaller than, but positively correlated with, $\partial h/\partial t$ is a somewhat restrictive one. For example, a standing-wave form $H(x)\cos\omega t$ for h, although it could always be made to satisfy (9) and (10) by incorporation of a linear 'recoil function' in $H(x)$, would give a Froude efficiency

$$\eta_F = 1 - \frac{1}{2}\frac{\omega^2 H^2(l) + U^2 H'^2(l)}{\omega^2 H^2(l)}, \tag{12}$$

which cannot exceed $\frac{1}{2}$. This is doubtless why fish do not normally attempt to swim by causing their bodies to execute the rather simple movements of a standing wave.

A more satisfactory form of $h(x,t)$, if only it could be made to satisfy (9) and (10), would be

$$h(x,t) = f(x)\,g\left(t-\frac{x}{c}\right), \tag{13}$$

where $g(t)$ is an oscillatory function such as $\cos\omega t$. Equation (13) represents a travelling wave, which moves down the fish's body with velocity c, and whose amplitude $f(x)$ may vary with position along the fish. Substitution in (8) gives

$$\begin{aligned}\eta_F &= 1-\frac{1}{2}\frac{\overline{\left\{\left(1-\frac{U}{c}\right)f(l)\,g'\left(t-\frac{l}{c}\right)+Uf'(l)\,g\left(t-\frac{l}{c}\right)\right\}^2}}{\overline{f(l)\,g'\left(t-\frac{l}{c}\right)\left\{\left(1-\frac{U}{c}\right)f(l)\,g'\left(t-\frac{l}{c}\right)+Uf'(l)\,g\left(t-\frac{l}{c}\right)\right\}}}\\ &= 1-\frac{1}{2}\frac{\left(1-\frac{U}{c}\right)^2 f^2(l)\,\overline{g'^2}+U^2f'^2(l)\,\overline{g^2}}{\left(1-\frac{U}{c}\right)f^2(l)\,\overline{g'^2}}.\end{aligned} \tag{14}$$

The rate of working is positive only if $c > U$. To make η_F close to 1 it is desirable to have $f'(l)$ practically zero; indeed, a non-zero value of $f'(l)$ reduces the thrust without altering the rate of working. On the other hand, it is wasteful to keep $f(x)$ constant all along the length of the fish, since the thrust depends only on values at $x = l$; and it will be found desirable, also for other reasons, to let $f(x)$ increase gradually from zero at $x = 0$ to a maximum, with $f'(l) = 0$, at the trailing edge.

Under these circumstances, η_F would be $\frac{1}{2}(1+U/c)$, and for example, $\eta_F = 0{\cdot}9$ for $c = 1{\cdot}25U$. Such a value of c is sufficiently in excess of U to give a substantial positive mean thrust

$$\overline{P} = \tfrac{1}{2}\rho A(l)\left(1-\frac{U^2}{c^2}\right)f^2(l)\,\overline{g'^2}. \tag{15}$$

To investigate whether the thrust (15) would be adequate with $c = 1{\cdot}25U$, note that in a real flow it would have to balance the frictional drag

$$D = \tfrac{1}{2}\rho U^2 C_D S, \tag{16}$$

where C_D is a drag coefficient based on the total surface area S. In (15), $f^2(l)\,\overline{g'^2}$ is the mean square lateral velocity of the trailing edge, which should not exceed $0{\cdot}05U^2$ if the assumptions which have been made in the theory are to hold—and, indeed, these restrictions to small disturbances are almost certainly beneficial in reducing drag. With this limitation, and with $c = 1{\cdot}25U$, the balance of thrust and drag can be achieved only if

$$A(l) > \frac{C_D S}{0{\cdot}018}. \tag{17}$$

Here $A(l) = \frac{1}{4}\pi s^2$, where s is the span of the tail trailing edge, and equation (17) indicates the advantage for efficiency of having a tail of adequate span.

Such a conclusion would be too optimistic, however, as it does not take account of recoil. A precise solution of the problem of what efficiency can be obtained with adequate thrust, and within the limitations set by our assumptions, from a displacement function into which a recoil function (11) has been incorporated to make it satisfy (9) and (10), would necessitate an extensive computing programme. However, the following general remarks would probably be supported by such computations.

Any angular recoil (represented by the second term in (11)) tends to reduce thrust without affecting rate of working, for it produces extra terms in $(\partial h/\partial x)_{x=l}$ which, being dependent on displacement rather than velocity, do not correlate well with $(\partial h/\partial t)_{x=l}$, and therefore contribute (through their mean square) to the numerator of (8) but not to the denominator—just as did the terms in $f'^2(l)$ in (14).

By contrast, a small transverse recoil of the mass-centre without angular motion, if well correlated with the trailing-edge movement, would not alter the efficiency that can be obtained for given mean square trailing-edge velocity and mean thrust, although it would somewhat increase the value of the wave-velocity c required to obtain that efficiency. (This is because the reduction of $\partial h/\partial t$ requires a corresponding reduction of $\partial h/\partial x$.)

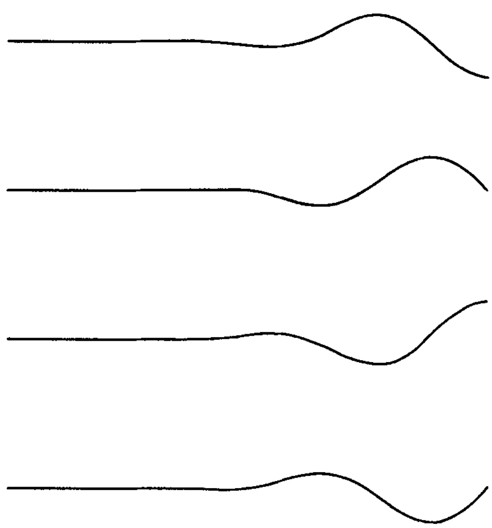

FIGURE 1. Suggested cycle of swimming movements. Successive shapes of fish centre-line are shown, with time increasing downwards. Motion is from right to left.

Accordingly, such travelling wave-forms as produce minimal angular recoil are desirable. A good solution appears to be as follows:

(i) confine motions to the rear part of the fish where fish mass is low;

(ii) keep V low for given $\partial h/\partial t$ by having c near to U, thus reducing water momentum in spite of the high virtual mass of the rear cross-sections;

(iii) most important of all, let the wave-form have a positive and a negative phase in the region of substantial amplitude (figure 1), so that the angular recoils produced by each tend to cancel out. (This, incidentally, will tend also to reduce the transverse recoil of the mass centre and so keep down the value of c, and hence of V, thus assisting (ii).)

These arguments seem to confirm that the type of swimming action favoured by most fish has good efficiency.

4. Boundary-layer considerations

The inviscid flow around the swimming fish, which is calculated in detail in the Appendix and whose effects on its dynamics have been studied in §§ 2 and 3, provides (as remarked in § 1) the environment in which the boundary layer develops. We now consider briefly the possible effects of the boundary layer on the flow and on the dynamics, asking first whether separation occurs and if so to what effect, and secondly whether there is transition to turbulence.

Separation is most likely to occur in the cross-flows. Consider a thin slab of fluid with plane faces perpendicular to the x-direction, moving past the fish at velocity U with only slight distortions from the plane shape. This slab of fluid is aware of the presence of a body of approximately cylindrical shape C_x in its midst, but the cross-section S_x of that body changes slowly with time and exhibits a movement through the fluid in the z-direction with velocity $V(x,t)$. If the flow produce by this movement has momentum M and energy E per unit width of slab, then in the absence of boundary-layer separation M and E have the potential-flow values ρVA and $\frac{1}{2}\rho V^2A$, and the resultant forces are as calculated in § 2.

When, however, vorticity is released from the surface by boundary-layer separation in this transverse flow, M and E are altered by amounts equal to the momentum and energy of the vortex system in the presence of the boundary. But the physical arguments of § 2 indicate that we will still have

$$\overline{W} = U\overline{\left\{M\frac{\partial h}{\partial t}\right\}}_{x=l}; \tag{18}$$

that is, the mean rate of working of the fish will be equal to the mean product of the lateral velocity of the trailing edge with the rate of shedding of momentum from it; while the wasted part of this rate of working will be

$$\overline{W}-\overline{P}U = U\{\overline{E}\}_{x=l}, \tag{19}$$

that is, the rate of energy shedding from the trailing edge. The Froude efficiency is therefore

$$\eta_F = 1-\frac{\{\overline{E}\}_{x=l}}{\{\overline{M\,\partial h/\partial t}\}_{x=l}}. \tag{20}$$

Now, it follows from (20) that, for the efficient types of swimming motion discussed at the end of § 3, the effect of η_F on vortex contributions to E and M is limited to a slight increase in the factor $\frac{1}{2}$ in equation (8). Note first that, even when the wave-form has a positive and a negative phase as in figure 1, the slab of fluid moving past it, at a velocity U only a little less than the wave velocity, experiences cross-section movements essentially in a single direction, with

V increasing to a maximum at the tail. Whatever vorticity is generated, the fluid energy per unit width of slab must increase during this process at a rate

$$\frac{dE}{dt} = V\frac{dM}{dt}. \tag{21}$$

The effect of vortex shedding as V increases means that M will increase more rapidly than in direct proportion to V, due to the vortex drag, and this with (21) makes E/MV somewhat greater than $\frac{1}{2}$ (for example, if $M \propto V^3$, then E/MV rises to $\frac{3}{4}$), although M remains in phase with V and E with V^2. These considerations make a change in the factor $\frac{1}{2}$ in equation (8), but do not qualitatively alter the conclusions about efficiency; for example, 85% is not much worse than 90% in this context.

Physically, this is still because transverse velocities V of fish relative to fluid are being kept small compared with absolute transverse velocities.

As to the boundary layer in the flow along the fish, the additional pressures due to the transverse motions (called p_1+p_2 in the Appendix) are too small, when V is kept small relative to $\partial h/\partial t$, to affect substantially its development. The only problem, therefore, is the long-standing one of why the boundary layer should have any special tendency to laminar flow. Here one can only draw attention to the remarkable results achieved by Coleman-Kramer Inc. of Los Angeles with their 'Lamiflo' coating (Judge 1960), which consists of a thin rubber skin attached to the surface of a body by short rubber pillars between which a free-flowing viscous fluid is present. This has been found to halve turbulent boundary-layer drag of the body in water, presumably owing to reduction of turbulence level by damping of surface pressure fluctuations. It is possible that layers of blubber can have a similar effect.

The author expresses his gratitude to Sir James Gray for extremely interesting discussions of the problem.

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Appendix. The inviscid flow around a slender fish

To obtain the pressure distribution in this problem it is almost essential to make a transformation of co-ordinates, so that the body becomes a fixed surface—for, otherwise, there are severe difficulties due to applying boundary conditions at a surface whose position is displaced in a direction in which gradients are specially steep.

Accordingly we introduce new co-ordinates X, Y, Z and T, where

$$X = x, \quad Y = y, \quad Z = z-h(x,t), \quad T = t \tag{A 1}$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - \frac{\partial h}{\partial x}\frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T} - \frac{\partial h}{\partial t}\frac{\partial}{\partial Z}. \tag{A 2}$$

Then Laplace's equation for the potential ϕ becomes

$$\left(\frac{\partial}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial}{\partial Z}\right)\left(\frac{\partial \phi}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial \phi}{\partial Z}\right)+\frac{\partial^2 \phi}{\partial Y^2}+\frac{\partial^2 \phi}{\partial Z^2}=0. \tag{A 3}$$

If now ϵ is a slenderness parameter (so that the fish's lateral dimensions do not exceed ϵl, nor its lateral velocities ϵU), then the first term in (A 3) is of order ϵ^2 relative to the second and third, so that near the body we can use the two-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial Y^2}+\frac{\partial^2 \phi}{\partial Z^2}=0. \tag{A 4}$$

If now the equation of the surface S of the stretched straight fish is $F(x,y,z)=0$, then in these co-ordinates (A 1) the surface of the swimming fish has the equation $F(X,Y,Z)=0$, and the boundary condition on it is obtained by setting equal to zero the rate of change DF/Dt of $F(X,Y,Z)$ following a particle of fluid. By (A 2), this gives

$$-\frac{\partial h}{\partial T}\frac{\partial F}{\partial Z}+\left(\frac{\partial \phi}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial \phi}{\partial Z}\right)\left(\frac{\partial F}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial F}{\partial Z}\right)+\frac{\partial \phi}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \phi}{\partial Z}\frac{\partial F}{\partial Z}=0. \tag{A 5}$$

We now put $\qquad \phi = UX+\phi_0(X,Y,Z)+\phi_1(X,Y,Z,T), \qquad$ (A 6)

where $UX+\phi_0$ is the potential of the flow when $h=0$ (that is, the steady flow when the fish is held stationary in the stream). For a slender fish $\partial F/\partial X$ is small compared with $\partial F/\partial Y$ and $\partial F/\partial Z$, and derivatives of ϕ_0 and ϕ_1 with respect to X, Y and Z are small compared with U. Also $\partial h/\partial X$ is small, and $\partial h/\partial T$ small compared with U. Hence, with products of small quantities neglected in (A 5), we obtain

$$-\frac{\partial h}{\partial T}\frac{\partial F}{\partial Z}+U\left(\frac{\partial F}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial F}{\partial Z}\right)+\left(\frac{\partial \phi_0}{\partial Y}+\frac{\partial \phi_1}{\partial Y}\right)\frac{\partial F}{\partial Y}+\left(\frac{\partial \phi_0}{\partial Z}+\frac{\partial \phi_1}{\partial Z}\right)\frac{\partial F}{\partial Z}=0. \tag{A 7}$$

The special case $h=0$, $\phi_1=0$ gives us

$$\frac{\partial \phi_0}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \phi_0}{\partial Z}\frac{\partial F}{\partial Z}=-U\frac{\partial F}{\partial X} \tag{A 8}$$

as the boundary condition for ϕ_0. Subtracting this from (A 7), we obtain

$$\frac{\partial \phi_1}{\partial Y}\frac{\partial F}{\partial Y}+\left(\frac{\partial \phi_1}{\partial Z}-\frac{\partial h}{\partial T}-U\frac{\partial h}{\partial X}\right)\frac{\partial F}{\partial Z}=0 \tag{A 9}$$

as the boundary condition for ϕ_1.

We shall not discuss the method of calculating the steady-flow potential ϕ_0, which is fairly well known, and leads to a pressure distribution with no resultant force or moment for a symmetrical shape like a fish. Passing therefore to ϕ_1, we note that equation (A 4), with the boundary condition (A 9) and the condition that $\phi_1 \to 0$ at infinity, implies that, for each X, ϕ_1 is the potential of the two-dimensional flow in the Y, Z plane, resulting from the movement of an infinite cylinder C_X, whose cross-section is S_X, that is, the fish cross-section for the given value of X, and which moves in the Z-direction with velocity

$$V(X,T)=\frac{\partial h}{\partial T}+U\frac{\partial h}{\partial X}. \tag{A 10}$$

If, therefore, $\Phi(X, Y, Z)$ is the potential due to the said cylinder C_X moving with unit velocity in the Z-direction through fluid at rest—note that it varies with X only because the choice of cylinder cross-section does—then the solution to the equations for ϕ_1 is

$$\phi_1(X, Y, Z, T) = V(X, T)\,\Phi(X, Y, Z). \tag{A 11}$$

We now calculate the associated pressure variations. Here we must remember that derivatives of ϕ_0 and ϕ_1 with respect to X are of order $U\epsilon^2$ (more strictly, $U\epsilon^2 \log \epsilon^{-1}$),* but that derivatives with respect to Y and Z are of order $U\epsilon$. From Bernoulli's equation

$$p-p_\infty = -\rho\left(\frac{\partial\phi}{\partial T}-\frac{\partial h}{\partial T}\frac{\partial\phi}{\partial Z}\right)-\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial\phi}{\partial Z}\right)^2-\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial Y}\right)^2-\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial Z}\right)^2+\tfrac{1}{2}U^2, \tag{A 12}$$

we must pick out the terms of order ϵ^2, neglecting those of order ϵ^4. This gives

$$\begin{aligned} p-p_\infty &= -\rho\frac{\partial\phi_1}{\partial T}+\rho\frac{\partial h}{\partial T}\left(\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_1}{\partial Z}\right)-\rho U\left(\frac{\partial\phi_0}{\partial X}+\frac{\partial\phi_1}{\partial X}\right)\\ &\qquad +\rho U\frac{\partial h}{\partial X}\left(\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_1}{\partial Z}\right)-\tfrac{1}{2}\rho\left(\frac{\partial\phi_0}{\partial Y}+\frac{\partial\phi_1}{\partial Y}\right)^2-\tfrac{1}{2}\rho\left(\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_1}{\partial Z}\right)^2\\ &= \rho\left\{-U\frac{\partial\phi_0}{\partial X}-\frac{1}{2}\left(\frac{\partial\phi_0}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial\phi_0}{\partial Z}\right)^2\right\}+\rho\left\{-\frac{\partial\phi_1}{\partial T}-U\frac{\partial\phi_1}{\partial X}\right.\\ &\qquad \left.+\left(V-\frac{\partial\phi_1}{\partial Z}\right)\frac{\partial\phi_0}{\partial Z}-\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_0}{\partial Y}\right\}+\rho\left\{V\frac{\partial\phi_1}{\partial Z}-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial Z}\right)^2\right\}\\ &= p_0+p_1+p_2, \end{aligned} \tag{A 13}$$

say, where p_0 is the pressure distribution in steady flow past the stretched straight fish, p_2 is the pressure distribution due to steady motion of the cylinder C_X through fluid at rest with velocity V, and p_1 is the remainder of the pressure distribution. Thus, p_0 depends solely (and quadratically) on the shape of the stretched straight fish, p_2 solely (and quadratically) on its displacements, and p_1 linearly on both.

Now we are interested mainly in the distribution of the lift (that is, resultant force in the z-direction) per unit length, $L(X, T)$, and in the resultant thrust P. We obtain L to order ϵ^3, neglecting terms of order ϵ^5.

There is no contribution to lift from p_0 or p_2; for p_0 has zero resultant in the z-direction over a cross-section by the definition of the stretched straight position, and p_2 has because the resistance to the motion of C_X in steady potential flow is zero. Therefore, only p_1 produces any local normal force, and by (A 13)

$$L = -\rho\oint_{S_X}\left\{\frac{\partial\phi_1}{\partial T}+U\frac{\partial\phi_1}{\partial X}+\left(\frac{\partial\phi_1}{\partial Z}-V\right)\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_0}{\partial Y}\right\}dY. \tag{A 14}$$

Now in (A 14) we can use the facts that $\phi_1 = V\Phi$, and that

$$\oint_{S_X}\Phi\,dY = A(X), \tag{A 15}$$

* For this reason the terms neglected in (A 7) were of order ϵ^2 (more strictly $\epsilon^2 \log \epsilon^{-1}$) relative to those retained.

where $\rho A(X)$ is the virtual mass per unit length of the cylinder C_X, whose cross-section is S_X, when moving in the z-direction; thus, $A(X)$ has the dimensions of area. From (A 15) it follows that

$$\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\{VA(X)\}$$
$$=\oint_{S_X}\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)(V\Phi)\,dY+U\oint_{S_X}V\frac{\partial\Phi}{\partial Z}\left(\frac{\partial Z}{\partial X}\right)_{Y\text{ const.}}dY, \quad \text{(A 16)}$$

where the last term results from the variation of the cross-section S_X with X, which causes the value of Z for given Y to vary along the surface.

On the right-hand side of (A 16), the first term can be identified with the first part of the integral in (A 14). Also, the second term can be identified with the second part, for it is

$$-U\oint_{S_X}\frac{\partial\phi_1}{\partial Z}\frac{\partial F/\partial X}{\partial F/\partial Z}\,dY, \quad \text{(A 17)}$$

which by (A 8) is

$$\oint_{S_X}\frac{\partial\phi_1}{\partial Z}\left(\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_0}{\partial Y}\frac{\partial F/\partial Y}{\partial F/\partial Z}\right)dY=\oint_{S_X}\frac{\partial\phi_1}{\partial Z}\left(\frac{\partial\phi_0}{\partial Z}dY-\frac{\partial\phi_0}{\partial Y}dZ\right)$$
$$=\iint_{E_X}\left\{\frac{\partial}{\partial Z}\left(\frac{\partial\phi_1}{\partial Z}\frac{\partial\phi_0}{\partial Z}\right)+\frac{\partial}{\partial Y}\left(\frac{\partial\phi_1}{\partial Z}\frac{\partial\phi_0}{\partial Y}\right)\right\}dY\,dZ, \quad \text{(A 18)}$$

where E_X is the area external to S_X. Using Laplace's equation in two dimensions for ϕ_0 and then for ϕ_1, we can write (A 18) as

$$\iint_{E_X}\left\{-\frac{\partial^2\phi_1}{\partial Y^2}\frac{\partial\phi_0}{\partial Z}+\frac{\partial^2\phi_1}{\partial Y\partial Z}\frac{\partial\phi_0}{\partial Y}\right\}dY\,dZ=\oint_{S_X}\left(\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_0}{\partial Z}dZ+\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_0}{\partial Y}dY\right)$$
$$=\oint_{S_X}\left\{\left(\frac{\partial\phi_1}{\partial Z}-V\right)\frac{\partial\phi_0}{\partial Z}+\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_0}{\partial Y}\right\}dY, \quad \text{(A 19)}$$

where the boundary condition (A 9) has been used to throw (A 17) into the form shown in the second part of (A 14).

This equation now shows, as stated in § 2, and there interpreted physically, that the lift per unit length is

$$L(X,T)=-\rho\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\{VA(X)\}, \quad \text{(A 20)}$$

where V is given by (A 10).

The mean value of the thrust P is now obtained to order ϵ^4, neglecting terms of order ϵ^6. Writing P as an integral over the surface of the swimming fish, and then using the co-ordinates (A 1) to express it as an integral over S, we obtain

$$P=\iint(p_0+p_1+p_2)\,dy\,dz=\iint_S(p_0+p_1+p_2)\,dY\left(\frac{\partial h}{\partial X}dX+dZ\right)$$
$$=\int_0^l L(X,T)\frac{\partial h}{\partial X}dX+\iint_S(p_0+p_1+p_2)\,dY\,dZ=P_1+P_2, \quad \text{(A 21)}$$

where P_1 is to be interpreted as the resultant in the forward direction of the lift forces $L(X, T)$, and we have simply

$$P_2 = \iint_S p_2 \, dY \, dZ, \tag{A 22}$$

since the pressures p_0 have zero resultant, and for a symmetrical fish $\iint p_1 \, dY \, dZ$ is zero (while even for an unsymmetrical fish the mean of this oscillatory force would be zero).

In (A 22), the $dY dZ$ is an elementary area of the surface S of the stretched straight fish, projected in the negative X-direction (or thrust direction). Such an area lying between cross-sections S_X and $S_{X+\delta X}$ can be written $-(\delta n)\, ds$, where ds is an element of length around S_X, and δn is the normal distance between the curves S_X and $S_{X+\delta X}$ when projected on to the same plane (figure 2). Accordingly, that part of P_2 which arises from between the cross-sections S_X and $S_{X+\delta X}$ can be written

$$-\oint_{S_X} (\delta n) \, p_2 \, ds. \tag{A 23}$$

Now, p_2 is the pressure distribution in potential flow due to the motion of the cylinder C_X through fluid with constant velocity V in the z-direction. In this

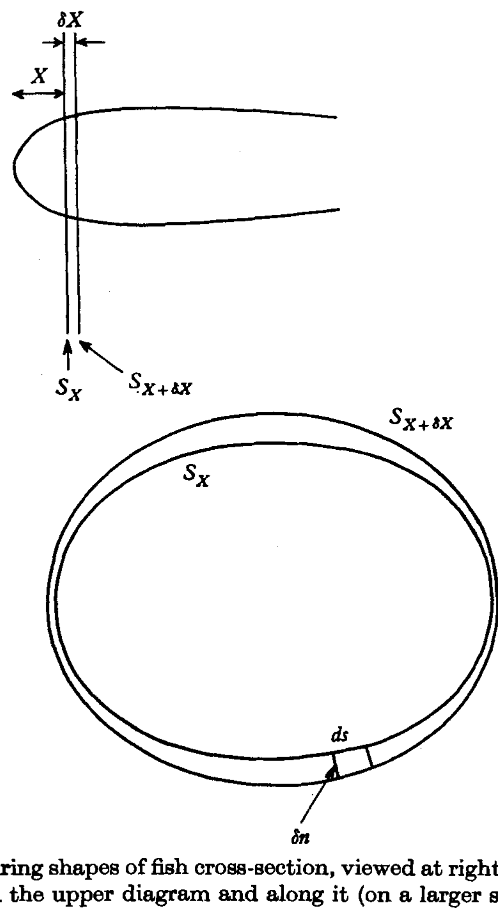

FIGURE 2. Neighbouring shapes of fish cross-section, viewed at right angles to the direction of locomotion in the upper diagram and along it (on a larger scale) in the lower.

motion the kinetic energy of the fluid per unit length of cylinder is $\frac{1}{2}\rho V^2 A(X)$ and the momentum $\rho V A(X)$. If now, during this motion, the cylinder C_X performed gradually a small change of cross-section from S_X to $S_{X+\delta X}$, the fluid momentum would change by

$$\rho V \delta A = \rho V\{A(X+\delta X)-A(X)\} = \rho V \frac{dA}{dX}\delta X, \tag{A 24}$$

and the cylinder, moving at constant speed V, must do work $\rho V^2 \delta A$ to produce this change of momentum.

But the motion of the cylinder is not simply rectilinear; in addition, each element ds of the surface moves outwards a distance δn (figure 2) against a pressure p_2. This means that the cylinder must also do a quantity of work given by the integral in (A 23), per unit length. But the total work done must equal the change in the kinetic energy $\frac{1}{2}\rho V^2 A$ of the fluid, giving*

$$\rho V^2 \delta A + \oint (\delta n)\, p_2\, ds = \tfrac{1}{2}\rho V^2 \delta A. \tag{A 25}$$

This evaluates (A 23), the part of P_2 which arises from between the cross-sections S_X and $S_{X+\delta X}$, and enables us to write

$$P_2 = \int_0^l \tfrac{1}{2}\rho V^2 dA(X). \tag{A 26}$$

We now throw the mean value of the part P_1 of the thrust, which arises from forward-directed lift, into a similar form. By (A 20) and (A 21), it is

$$\begin{aligned} P_1 &= -\rho \int_0^l \frac{\partial h}{\partial X}\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\{VA(X)\}\,dX \\ &= -\rho\int_0^l \left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\left\{\frac{\partial h}{\partial X}VA(X)\right\}dX + \rho\int_0^l \frac{\partial V}{\partial X}VA(X)\,dX. \end{aligned} \tag{A 27}$$

The mean over a long time of the time-derivative in this last expression is zero. Hence the mean value $\overline{P_1}$ is

$$\overline{P_1} = -\rho U\left[\overline{\frac{\partial h}{\partial X}VA}\right]_0^l + \rho[\tfrac{1}{2}\overline{V^2}A]_0^l - \tfrac{1}{2}\rho\int_0^l \overline{V^2}\,dA(X). \tag{A 28}$$

Adding this to the averaged form of (A 26), we obtain, for the total mean thrust $\overline{P} = \overline{P_1}+\overline{P_2}$, the simple equation

$$\overline{P} = \tfrac{1}{2}\rho A(l)\left\{\overline{\left(\frac{\partial h}{\partial T}\right)^2} - U^2\overline{\left(\frac{\partial h}{\partial X}\right)^2}\right\}, \tag{A 29}$$

which was stated in § 2 as equation (7).

* The correctness of this formula is easily checked for a circle or an ellipse. When the cross-section is increasing, the integral in (A 23) comes out negative, which physically is because the regions of suction outweigh the regions of pressure around the surface.